# STRUCTURE OF THE EXTENDED SCHRÖDINGER-VIRASORO LIE ALGEBRA $\widetilde{\mathfrak{sv}}$*

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ABSTRACT. In this paper, we study the derivations, the central extensions and the automorphism group of the extended Schrödinger-Virasoro Lie algebra $\widetilde{\mathfrak{sv}}$, introduced by J. Unterberger [25] in the context of two-dimensional conformal field theory and statistical physics. Moreover, we show that $\widetilde{\mathfrak{sv}}$ is an infinite-dimensional complete Lie algebra and the universal central extension of $\widetilde{\mathfrak{sv}}$ in the category of Leibniz algebras is the same as that in the category of Lie algebras.

## 1. Introduction

The Schrödinger-Virasoro Lie algebra $\mathfrak{sv}$, originally introduced by M. Henkel in [8] during his study on the invariance of the free Schrödinger equation, is a vector space over the complex field $\mathbb{C}$ with a basis $\{L_n, M_n, Y_{n+\frac{1}{2}} \mid n \in \mathbb{Z}\}$ and the Lie brackets:

$$[L_m, L_n] = (n-m)L_{m+n}, \quad [L_m, M_n] = nM_{m+n}, \quad [L_m, Y_{n+\frac{1}{2}}] = (n + \frac{1-m}{2})Y_{m+n+\frac{1}{2}},$$
$$[M_m, M_n] = 0, \quad [Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}] = (m-n)M_{m+n+1}, \quad [M_m, Y_{n+\frac{1}{2}}] = 0,$$

for all $m, n \in \mathbb{Z}$. It is easy to see that $\mathfrak{sv}$ is a semi-direct product of the centerless Virasoro algebra (Witt algebra) $\mathfrak{Vir}_0 = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n$, which can be regarded as the Lie algebra that consists of derivations on the Laurent polynomial ring [14], and the two-step nilpotent infinite-dimensional Lie algebra $\mathfrak{h} = \bigoplus_{m\in\mathbb{Z}} \mathbb{C}Y_{m+\frac{1}{2}} \bigoplus_{m\in\mathbb{Z}} \mathbb{C}M_m$, which contains the Schrödinger Lie algebra $\mathfrak{s}$ spanned by $\{L_{-1}, L_0, L_1, Y_{-\frac{1}{2}}, Y_{\frac{1}{2}}, M_0\}$. Clearly $\mathfrak{s}$ is isomorphic to the semi-direct product of the Lie algebra $\mathfrak{sl}(2)$ and the three-dimensional nilpotent Heisenberg Lie algebra $\langle Y_{-\frac{1}{2}}, Y_{\frac{1}{2}}, M_0\rangle$. The structure and representation theory of $\mathfrak{sv}$ have been extensively studied by C. Roger and J. Unterberger. We refer the reader to [22] for more details. Recently, in order to investigate vertex representations of $\mathfrak{sv}$, J. Unterberger [25] introduced a class of new infinite-dimensional Lie algebras $\widetilde{\mathfrak{sv}}$ called the extended Schrödinger-Virasoro algebra (see section 2), which can be viewed as an extension of $\mathfrak{sv}$ by a conformal current with conformal weight 1.

**Keywords: Schrödinger-Virasoro algebra, central extension, derivation, automorphism**.

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In this paper, we give a complete description of the derivations, the central extensions and the automorphisms for the extended Schrödinger-Virasoro Lie algebra $\widetilde{\mathfrak{sv}}$. The paper is organized as follows:

In section 2, we show that the center of $\widetilde{\mathfrak{sv}}$ is zero and all derivations of $\widetilde{\mathfrak{sv}}$ are inner derivations, i.e., $\mathrm{H}^1(\widetilde{\mathfrak{sv}}, \widetilde{\mathfrak{sv}}) = 0$, which implies that $\widetilde{\mathfrak{sv}}$ is a complete Lie algebra. Recall that a Lie algebra is called complete if its center is zero and all derivations are inner, which was originally introduced by N. Jacobson in [10]. Over the past decades, much progress has been obtained on the theory of complete Lie algebras ( see for example [13, 26, 20]). Note that since the center of $\mathfrak{sv}$ is non-zero and dim $\mathrm{H}^1(\mathfrak{sv}, \mathfrak{sv}) = 3$, $\mathfrak{sv}$ is not a complete Lie algebra.

In section 3, we determine the universal central extension of $\widetilde{\mathfrak{sv}}$. The universal covering algebra $\widehat{\mathfrak{sv}}$ contains the twisted Heisenberg-Virasoro Lie algebra, which plays an important role in the representation theory of toroidal Lie algebras [2, 3, 11, 24], as a subalgebra. Furthermore, in section 4, we show that there is no non-zero symmetric invariant bilinear form on $\widetilde{\mathfrak{sv}}$, which implies the universal central extension of $\widetilde{\mathfrak{sv}}$ in the category of Leibniz algebras is the same as that in the category of Lie algebras [9].

Finally in section 5, we give the automorphism groups of $\widetilde{\mathfrak{sv}}$ and its universal covering algebra $\widehat{\mathfrak{sv}}$, which are isomorphic.

Throughout the paper, we denote by $\mathbb{Z}$ and $\mathbb{C}^*$ the set of integers and the set of non-zero complex numbers respectively, and all the vector spaces are assumed over the complex field $\mathbb{C}$.

## 2. The Derivation Algebra of $\widetilde{\mathfrak{sv}}$

**Definition 2.1.** *The extended Schrödinger-Virasoro Lie algebra $\widetilde{\mathfrak{sv}}$ is a vector space spanned by a basis $\{L_n, M_n, N_n, Y_{n+\frac{1}{2}} \mid n \in \mathbb{Z}\}$ with the following brackets*

$$[L_m, L_n] = (n-m)L_{m+n},\ \ [M_m, M_n] = 0,\ \ [N_m, N_n] = 0,\ \ [Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}] = (m-n)M_{m+n+1},$$

$$[L_m, M_n] = nM_{m+n},\quad [L_m, N_n] = nN_{m+n},\quad [L_m, Y_{n+\frac{1}{2}}] = (n + \frac{1-m}{2})Y_{m+n+\frac{1}{2}},$$

$$[N_m, M_n] = 2M_{m+n},\quad [N_m, Y_{n+\frac{1}{2}}] = Y_{m+n+\frac{1}{2}},\quad [M_m, Y_{n+\frac{1}{2}}] = 0,$$

*for all $m, n \in \mathbb{Z}$.*

It is clear that $\widetilde{\mathfrak{sv}}$ is a perfect Lie algebra, i.e., $[\widetilde{\mathfrak{sv}}, \widetilde{\mathfrak{sv}}] = \widetilde{\mathfrak{sv}}$ , which is finitely generated with a set of generators $\{L_{-2}, L_{-1}, L_1, L_2, N_1, Y_{\frac{1}{2}}\}$.

Define a $\frac{1}{2}\mathbb{Z}$-grading on $\widetilde{\mathfrak{sv}}$ by

$$deg(L_n) = n,\ deg(M_n) = n,\ deg(N_n) = n,\ deg(Y_{n+\frac{1}{2}}) = n + \frac{1}{2},$$

for all $n \in \mathbb{Z}$. Then

$$\widetilde{\mathfrak{sv}} = \bigoplus_{n\in\mathbb{Z}} \widetilde{\mathfrak{sv}}_{\frac{n}{2}} = (\bigoplus_{n\in\mathbb{Z}} \widetilde{\mathfrak{sv}}_n) \bigoplus (\bigoplus_{n\in\mathbb{Z}} \widetilde{\mathfrak{sv}}_{n+\frac{1}{2}}),$$

where $\widetilde{\mathfrak{sv}}_n = span\{L_n, M_n, N_n\}$ and $\widetilde{\mathfrak{sv}}_{n+\frac{1}{2}} = span\{Y_{n+\frac{1}{2}}\}$ for all $n \in \mathbb{Z}$.

**Definition 2.2.** ([5]) *Let $G$ be a commutative group, $\mathfrak{g} = \bigoplus_{g\in G} \mathfrak{g}_g$ a $G$-graded Lie algebra. A $\mathfrak{g}$-module $V$ is called $G$-graded, if*

$$V = \bigoplus_{g\in G} V_{g\in G}, \quad \mathfrak{g}_g V_h \subseteq V_{g+h}, \quad \forall\, g, h \in G.$$

**Definition 2.3.** ([5]) *Let $\mathfrak{g}$ be a Lie algebra and $V$ a $\mathfrak{g}$-module. A linear map $D : \mathfrak{g} \longrightarrow V$ is called a derivation, if for any $x, y \in \mathfrak{g}$, we have*

$$D[x, y] = x.D(y) - y.D(x).$$

*If there exists some $v \in V$ such that $D : x \mapsto x.v$, then $D$ is called an inner derivation.*

Let $\mathfrak{g}$ be a Lie algebra, $V$ a module of $\mathfrak{g}$. Denote by $Der(\mathfrak{g}, V)$ the vector space of all derivations, $Inn(\mathfrak{g}, V)$ the vector space of all inner derivations. Set

$$H^1(\mathfrak{g}, V) = Der(\mathfrak{g}, V)/Inn(\mathfrak{g}, V).$$

Denote by $Der(\mathfrak{g})$ the derivation algebra of $\mathfrak{g}$, $Inn(\mathfrak{g})$ the vector space of all inner derivations of $\mathfrak{g}$. We will prove that all the derivations of $\widetilde{\mathfrak{sv}}$ are inner derivations.

By Proposition 1.1 in [5], we have the following lemma.

**Lemma 2.4.**

$$Der(\widetilde{\mathfrak{sv}}) = \bigoplus_{n\in\mathbb{Z}} Der(\widetilde{\mathfrak{sv}})_{\frac{n}{2}},$$

*where $Der(\widetilde{\mathfrak{sv}})_{\frac{n}{2}}(\widetilde{\mathfrak{sv}}_{\frac{m}{2}}) \subseteq \widetilde{\mathfrak{sv}}_{\frac{m+n}{2}}$ for all $m, n \in \mathbb{Z}$.*

□

**Lemma 2.5.** $H^1(\widetilde{\mathfrak{sv}}_0, \widetilde{\mathfrak{sv}}_{\frac{n}{2}}) = 0, \quad \forall\, n \in \mathbb{Z} \setminus \{0\}.$

*Proof.* We have to prove

$$H^1(\widetilde{\mathfrak{sv}}_0, \widetilde{\mathfrak{sv}}_n) = 0, \ \ \forall n \in \mathbb{Z} \setminus \{0\},$$
$$H^1(\widetilde{\mathfrak{sv}}_0, \widetilde{\mathfrak{sv}}_{n+\frac{1}{2}}) = 0, \ \ \forall n \in \mathbb{Z}.$$

(1) For $m \neq 0$, let $\varphi : \widetilde{\mathfrak{sv}}_0 \longrightarrow \widetilde{\mathfrak{sv}}_m$ be a derivation. Assume that

$$\varphi(L_0) = a_1 L_m + b_1 M_m + c_1 N_m,$$
$$\varphi(M_0) = a_2 L_m + b_2 M_m + c_2 N_m,$$
$$\varphi(N_0) = a_3 L_m + b_3 M_m + c_3 N_m,$$

where $a_i, b_i, c_i \in \mathbb{C}, i = 1, 2, 3$. Since

$$\varphi[L_0, M_0] = [\varphi(L_0), M_0] + [L_0, \varphi(M_0)],$$

we have

$$a_2 m L_m + (b_2 m + 2c_1) M_m + c_2 m N_m = 0.$$

So $a_2 = 0, c_1 = -\dfrac{1}{2}b_2m, c_2 = 0$ and $\varphi(M_0) = b_2M_m$. Since

$$\varphi[L_0, N_0] = [\varphi(L_0), N_0] + [L_0, \varphi(N_0)],$$

we have

$$a_3mL_m + (b_3m - 2b_1)M_m + c_3mN_m = 0.$$

Then $a_3 = 0, b_1 = \dfrac{1}{2}b_3m, c_3 = 0$ and $\varphi(N_0) = b_3M_m$. Therefore, we get

$$\varphi(L_0) = a_1L_m + \frac{1}{2}b_3mM_m - \frac{1}{2}b_2mN_m, \quad \varphi(M_0) = b_2M_m, \quad \varphi(N_0) = b_3M_m.$$

Let $X_m = \dfrac{a_1}{m}L_m + \dfrac{1}{2}b_3M_m - \dfrac{1}{2}b_2N_m$, we have

$$\varphi(L_0) = [L_0, X_m], \quad \varphi(M_0) = [M_0, X_m], \quad \varphi(N_0) = [N_0, X_m].$$

Then $\varphi \in Inn(\widetilde{\mathfrak{sv}}_0, \widetilde{\mathfrak{sv}}_m)$. Therefore,

$$H^1(\widetilde{\mathfrak{sv}}_0, \widetilde{\mathfrak{sv}}_m) = 0, \quad \forall\, m \in \mathbb{Z} \setminus \{0\}.$$

(2) For all $m \in \mathbb{Z}$, let $\varphi : \widetilde{\mathfrak{sv}}_0 \longrightarrow \widetilde{\mathfrak{sv}}_{m+\frac{1}{2}}$ be a derivation. Assume that

$$\varphi(L_0) = aY_{m+\frac{1}{2}}, \quad \varphi(M_0) = bY_{m+\frac{1}{2}}, \quad \varphi(N_0) = cY_{m+\frac{1}{2}},$$

for some $a, b, c \in \mathbb{C}$. Because $\varphi[L_0, M_0] = [\varphi(L_0), M_0] + [L_0, \varphi(M_0)]$, we have

$$0 = [L_0, \varphi(M_0)] = [L_0, bY_{m+\frac{1}{2}}] = b(m + \frac{1}{2})Y_{m+\frac{1}{2}}.$$

So $b = 0$ and $\varphi(M_0) = 0$. Since $\varphi[L_0, N_0] = [\varphi(L_0), N_0] + [L_0, \varphi(N_0)]$, we have

$$0 = [aY_{m+\frac{1}{2}}, N_0] + [L_0, cY_{m+\frac{1}{2}}] = (c(m + \frac{1}{2}) - a)Y_{m+\frac{1}{2}}.$$

Then $a = c(m + \frac{1}{2})$. Hence, we get

$$\varphi(L_0) = c(m + \frac{1}{2})Y_{m+\frac{1}{2}}, \quad \varphi(M_0) = 0, \quad \varphi(N_0) = cY_{m+\frac{1}{2}}.$$

Letting $X_{m+\frac{1}{2}} = cY_{m+\frac{1}{2}}$, we obtain

$$\varphi(L_0) = [L_0, X_{m+\frac{1}{2}}], \quad \varphi(M_0) = [M_0, X_{m+\frac{1}{2}}], \quad \varphi(N_0) = [N_0, X_{m+\frac{1}{2}}].$$

Then $\varphi \in Inn(\widetilde{\mathfrak{sv}}_0, \widetilde{\mathfrak{sv}}_{m+\frac{1}{2}})$. Therefore,

$$H^1(\widetilde{\mathfrak{sv}}_0, \widetilde{\mathfrak{sv}}_{m+\frac{1}{2}}) = 0, \quad \forall\, m \in \mathbb{Z}.$$

$\square$

**Lemma 2.6.** *$Hom_{\widetilde{\mathfrak{sv}}_0}(\widetilde{\mathfrak{sv}}_{\frac{m}{2}}, \widetilde{\mathfrak{sv}}_{\frac{n}{2}}) = 0$ for all $m, n \in \mathbb{Z}$, $m \neq n$.*

*Proof.* Let $f \in Hom_{\widetilde{\mathfrak{sv}}_0}(\widetilde{\mathfrak{sv}}_{\frac{m}{2}}, \widetilde{\mathfrak{sv}}_{\frac{n}{2}})$, where $m \neq n$. Then for any $E_0 \in \widetilde{\mathfrak{sv}}_0, E_{\frac{m}{2}} \in \widetilde{\mathfrak{sv}}_{\frac{m}{2}}$, we have

$$f([E_0, E_{\frac{m}{2}}]) = [E_0, f(E_{\frac{m}{2}})].$$

Then $f([L_0, E_{\frac{m}{2}}]) = [L_0, f(E_{\frac{m}{2}})]$, i.e.,

$$\frac{m}{2} f(E_{\frac{m}{2}}) = [L_0, f(E_{\frac{m}{2}})] = \frac{n}{2} f(E_{\frac{m}{2}}).$$

So we have $f(E_{\frac{m}{2}}) = 0$ for all $m \neq n$. Therefore, we have $f = 0$. □

By Lemma 2.5-2.6 and Proposition 1.2 in [5], we have the following Lemma.

**Lemma 2.7.** $Der(\widetilde{\mathfrak{sv}}) = Der(\widetilde{\mathfrak{sv}})_0 + Inn(\widetilde{\mathfrak{sv}})$. □

**Lemma 2.8.** *For any $D \in Der(\widetilde{\mathfrak{sv}})_0$, there exist some $a, b, c \in \mathbb{C}$ such that*

$$D = ad(aL_0 - \frac{c}{2} M_0 + (b - \frac{a}{2}) N_0).$$

*Therefore, $Der(\widetilde{\mathfrak{sv}})_0 \subseteq Inn(\widetilde{\mathfrak{sv}})$.*

*Proof.* For any $D \in Der(\widetilde{\mathfrak{sv}})_0$, assume that for all $m \in \mathbb{Z}$,

$$\begin{aligned} D(L_m) &= a_{11}^m L_m + a_{12}^m M_m + a_{13}^m N_m, \\ D(M_m) &= a_{21}^m L_m + a_{22}^m M_m + a_{23}^m N_m, \\ D(N_m) &= a_{31}^m L_m + a_{32}^m M_m + a_{33}^m N_m, \\ D(Y_{m+\frac{1}{2}}) &= b^{m+\frac{1}{2}} Y_{m+\frac{1}{2}}, \end{aligned}$$

where $a_{ij}^m,\ b^{m+\frac{1}{2}} \in \mathbb{C},\ i, j = 1, 2, 3$. For any $E_{\frac{m}{2}} \in \widetilde{\mathfrak{sv}}_{\frac{m}{2}},\ E_{\frac{n}{2}} \in \widetilde{\mathfrak{sv}}_{\frac{n}{2}}$, we have

$$D[E_{\frac{m}{2}}, E_{\frac{n}{2}}] = [D(E_{\frac{m}{2}}), E_{\frac{n}{2}}] + [E_{\frac{m}{2}}, D(E_{\frac{n}{2}})].$$

In particular, $D[L_0, E_{\frac{m}{2}}] = [D(L_0), E_{\frac{m}{2}}] + [L_0, D(E_{\frac{m}{2}})]$. Then

$$\frac{m}{2} D(E_{\frac{m}{2}}) = [D(L_0), E_{\frac{m}{2}}] + \frac{m}{2} D(E_{\frac{m}{2}}).$$

So

$$[D(L_0), E_{\frac{m}{2}}] = 0.$$

Since $[D(L_0), L_1] = 0, [D(L_0), M_0] = 0$ and $[D(L_0), N_0] = 0$, we can deduce that

$$a_{11}^0 = a_{12}^0 = a_{13}^0 = 0.$$

So $D(L_0) = 0$. Because $D[M_0, L_m] = [D(M_0), L_m] + [M_0, D(L_m)]$, we have

$$a_{21}^0 m L_m - 2a_{13}^m M_m = 0.$$

Then

$$a_{21}^0 = 0,\ a_{13}^m = 0,\ D(M_0) = a_{22}^0 M_0 + a_{23}^0 N_0.$$

By the fact that $D[M_0, N_m] = [D(M_0), N_m] + [M_0, D(N_m)]$, we have

$$a_{21}^m L_m + a_{22}^m M_m + a_{23}^m N_m = (a_{22}^0 + a_{33}^m) M_m.$$

Therefore,

$$a_{21}^m = 0, \quad a_{22}^m = a_{22}^0 + a_{33}^m, \quad a_{23}^m = 0.$$

Then

$$a_{33}^0 = 0, \quad D(M_m) = a_{22}^m M_m = (a_{22}^0 + a_{33}^m) M_m.$$

Since $D[N_0, L_m] = [D(N_0), L_m] + [N_0, D(L_m)]$, we have

$$a_{31}^0 m L_m + 2a_{12}^m M_m = 0.$$

Then $a_{31}^0 m = 0, 2a_{12}^m = 0$ for all $m \in \mathbb{Z}$. Therefore,

$$a_{31}^0 = 0, \quad a_{12}^m = 0, \quad D(N_0) = a_{32}^0 M_0.$$

According to $D[N_m, N_n] = [D(N_m), N_n] + [N_m, D(N_n)]$, we obtain

$$(a_{31}^m n - a_{31}^n m) N_{m+n} + 2(a_{32}^n - a_{32}^m) M_{m+n} = 0.$$

Then $a_{31}^m n = a_{31}^n m, a_{32}^n = a_{32}^m$ for all $m, n \in \mathbb{Z}$. Hence, we have

$$a_{31}^m = a_{31}^1 m, \quad a_{32}^m = a_{32}^0, \quad \forall\, m \in \mathbb{Z}.$$

By the following two relations

$$\begin{aligned} &D[Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}] = [D(Y_{m+\frac{1}{2}}), Y_{n+\frac{1}{2}}] + [Y_{m+\frac{1}{2}}, D(Y_{n+\frac{1}{2}})], \\ &D[L_m, L_n] = [D(L_m), L_n] + [L_m, D(L_n)], \end{aligned}$$

we have

$$a_{22}^{m+n+1} = b^{m+\frac{1}{2}} + b^{n+\frac{1}{2}}, \qquad m \neq n. \tag{2.1}$$

$$a_{11}^{m+n} = a_{11}^m + a_{11}^n, \qquad m \neq n. \tag{2.2}$$

It is easy to see that $a_{11}^{-m} = -a_{11}^m$ for all $m \in \mathbb{Z}$. Let $n = 1$ in (2.2), then

$$a_{11}^{m+1} = a_{11}^m + a_{11}^1, \qquad m \neq 1.$$

By induction on $m \in \mathbb{Z}^+$ and $m \geq 3$, we have

$$a_{11}^m = a_{11}^2 + (m-2) a_{11}^1, \qquad m \geq 3.$$

Let $m = 4, n = -2$ in (2.2), then we have $a_{11}^2 = 2a_{11}^1$. Therefore,

$$a_{11}^m = m a_{11}^1, \qquad \forall\, m \in \mathbb{Z}.$$

Since $D[L_m, M_n] = [D(L_m), M_n] + [L_m, D(M_n)]$ and

$$D[L_m, N_n] = [D(L_m), N_n] + [L_m, D(N_n)],$$

we get

$$a_{22}^{m+n} = m a_{11}^1 + a_{22}^n, \qquad n \neq 0, \tag{2.3}$$

$$a_{31}^1 (m+n) = a_{31}^1 (n-m), \quad a_{33}^{m+n} = m a_{11}^1 + a_{33}^n, \quad n \neq 0.$$

Then we can deduce that

$$a_{31}^1 = 0, \quad a_{33}^{m+1} = m a_{11}^1 + a_{33}^1.$$

Note $a_{33}^0=0$, then $a_{11}^1=a_{33}^1$. Therefore,

$$a_{33}^m = ma_{11}^1, \quad a_{22}^m = a_{22}^0 + ma_{11}^1, \qquad \forall\, m\in\mathbb{Z}.$$

Because $D[L_m, Y_{n+\frac{1}{2}}] = [D(L_m), Y_{n+\frac{1}{2}}] + [L_m, D(Y_{n+\frac{1}{2}})]$, we have

$$b^{m+n+\frac{1}{2}} = ma_{11}^1 + b^{n+\frac{1}{2}}, \quad n+\frac{1-m}{2}\neq 0.$$

Let $m=1$, then

$$b^{n+1+\frac{1}{2}} = a_{11}^1 + b^{n+\frac{1}{2}}, \quad n\neq 0.$$

By (2.1), we get

$$a_{22}^0 + (m+n+1)a_{11}^1 = b^{m+\frac{1}{2}} + b^{n+\frac{1}{2}}, \qquad m\neq\ n. \tag{2.4}$$

Let $m=0, n=1$ in (2.4), then we have

$$a_{22}^0 + a_{11}^1 = 2b^{\frac{1}{2}}.$$

Let $n=0$ in (2.4), then we obtain $b^{m+\frac{1}{2}} = b^{\frac{1}{2}} + ma_{11}^1$ for all $m\in\mathbb{Z}$. Set $a_{11}^1=a, b^{\frac{1}{2}}=b, a_{32}^0=c$, then $a_{22}^0=2b-a$ and

$$\begin{aligned} &D(L_m) = maL_m, && D(M_m) = (2b-a+ma)M_m,\\ &D(N_m) = cM_m + maN_m, && D(Y_{m+\frac{1}{2}}) = (b+ma)Y_{m+\frac{1}{2}}, \end{aligned}$$

for all $m\in\mathbb{Z}$. Then we can deduce that

$$D = ad(aL_0 - \frac{c}{2}M_0 + (b-\frac{a}{2})N_0).$$

□

From the above lemmas, we obtain the following theorem.

**Theorem 2.9.** *$Der(\widetilde{\mathfrak{sv}}) = Inn(\widetilde{\mathfrak{sv}})$, i.e., $\mathrm{H}^1(\widetilde{\mathfrak{sv}}, \widetilde{\mathfrak{sv}})=0$.*

**Lemma 2.10.** *$C(\widetilde{\mathfrak{sv}})=\{0\}$, where $C(\widetilde{\mathfrak{sv}})$ is the center of $\widetilde{\mathfrak{sv}}$.*

*Proof.* For any $E_{\frac{n}{2}}\in\widetilde{\mathfrak{sv}}_{\frac{n}{2}}$, we have

$$[L_0, E_{\frac{n}{2}}] = \frac{n}{2}E_{\frac{n}{2}}.$$

It forces $x\in\widetilde{\mathfrak{sv}}_0$, for any $x\in C(\widetilde{\mathfrak{sv}})$, since $[L_0,x]=0$. Let $x=aL_0+bN_0+cM_0$, where $a,b,c\in\mathbb{C}$. Then

$$[x, L_1] = [aL_0+bN_0+cM_0, L_1] = [aL_0, L_1] = aL_1 = 0.$$

So $a=0$. By the following relations,

$$[x, Y_{\frac{1}{2}}] = [bN_0+cM_0, Y_{\frac{1}{2}}] = [bN_0, Y_{\frac{1}{2}}] = bY_{\frac{1}{2}} = 0,$$
$$[x, N_0] = [cM_0, N_0] = -2cM_0 = 0,$$

we have $b=0$ and $c=0$. Therefore, $x=0$. □

By Lemma 2.10 and Theorem 3.1, we have

**Corollary 2.11.** $\widetilde{\mathfrak{sv}}$ *is an infinite-dimensional complete Lie algebra.*

## 3. The Universal Central Extension of $\widetilde{\mathfrak{sv}}$

In this section, we discuss the structure of the universal central extension of $\widetilde{\mathfrak{sv}}$. Let us first recall some basic concepts. Let $\mathfrak{g}$ be a Lie algebra. A bilinear function $\psi : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$ is called a 2-cocycle on $\mathfrak{g}$ if for all $x, y, z \in \mathfrak{g}$, the following two conditions are satisfied:

$$\begin{aligned} &\psi(x, y) = -\psi(y, x), \\ &\psi([x, y], z) + \psi([y, z], x) + \psi([z, x], y) = 0. \end{aligned} \tag{3.1}$$

For any linear function $f : \mathfrak{g} \longrightarrow \mathbb{C}$, one can define a 2-cocycle $\psi_f$ as follows

$$\psi_f(x, y) = f([x, y]), \qquad \forall\, x, y \in \mathfrak{g}.$$

Such a 2-cocycle is called a 2-coboudary on $\mathfrak{g}$. Denote by $C^2(\mathfrak{g}, \mathbb{C})$ the vector space of 2-cocycles on $\mathfrak{g}$, $B^2(\mathfrak{g}, \mathbb{C})$ the vector space of 2-coboundaries on $\mathfrak{g}$. Then the quotient space $H^2(\mathfrak{g}, \mathbb{C}) = C^2(\mathfrak{g}, \mathbb{C})/B^2(\mathfrak{g}, \mathbb{C})$ is called the second cohomology group of $\mathfrak{g}$.

**Theorem 3.1.** $\dim H^2(\widetilde{\mathfrak{sv}}, \mathbb{C}) = 3$.

*Proof.* Let $\varphi : \widetilde{\mathfrak{sv}} \times \widetilde{\mathfrak{sv}} \longrightarrow \mathbb{C}$ be a 2-cocycle on $\widetilde{\mathfrak{sv}}$. Let $f : \widetilde{\mathfrak{sv}} \longrightarrow \mathbb{C}$ be a linear function defined by

$$\begin{aligned} &f(L_0) = -\frac{1}{2}\varphi(L_1, L_{-1}), \quad f(L_m) = \frac{1}{m}\varphi(L_0, L_m),\ m \neq 0, \\ &f(M_0) = -\varphi(L_1, M_{-1}), \quad f(M_m) = \frac{1}{m}\varphi(L_0, M_m),\ m \neq 0, \\ &f(N_0) = -\varphi(L_1, N_{-1}), \quad f(N_m) = \frac{1}{m}\varphi(L_0, N_m),\ m \neq 0, \\ &f(Y_{m+\frac{1}{2}}) = \frac{1}{m + \frac{1}{2}}\varphi(L_0, Y_{m+\frac{1}{2}}), \qquad \forall\, m \in \mathbb{Z}. \end{aligned}$$

Let $\overline{\varphi} = \varphi - \varphi_f$, where $\varphi_f$ is the 2-coboundary induced by $f$, then

$$\overline{\varphi}(x, y) = \varphi(x, y) - f([x, y]), \quad \forall\, x, y \in \widetilde{\mathfrak{sv}}.$$

By the known result on the central extension of the classical Witt algebra ( see [1] or [15] ), we have

$$\overline{\varphi}(L_m, L_n) = \alpha\delta_{m+n,0}(m^3 - m), \quad \forall\, m, n \in \mathbb{Z},\ \alpha \in \mathbb{C}.$$

By the fact that $\varphi([L_0, L_m], M_n) + \varphi([L_m, M_n], L_0) + \varphi([M_n, L_0], L_m) = 0$, we get

$$(m + n)\varphi(L_m, M_n) = n\varphi(L_0, M_{m+n}).$$

So

$$\varphi(L_m, M_n) = \frac{n}{m + n}\varphi(L_0, M_{m+n}), \quad m + n \neq 0. \tag{3.2}$$

Then it is easy to deduce that

$$\overline{\varphi}(L_m, M_n) = 0, \qquad m + n \neq 0.$$

Furthermore,

$$\overline{\varphi}(L_m, M_{-m}) = \varphi(L_m, M_{-m}) + mf(M_0) = \varphi(L_m, M_{-m}) - m\varphi(L_1, M_{-1}). \tag{3.3}$$

Similarly, denote $\overline{\varphi}(L_m, N_{-m}) = c(m)$ for all $m \in \mathbb{Z}$, then we have

$$\overline{\varphi}(L_m, N_n) = \delta_{m+n,0}c(m), \quad \overline{\varphi}(L_m, N_{-m}) = \varphi(L_m, N_{-m}) - m\varphi(L_1, N_{-1}). \tag{3.4}$$

By (3.1), $\varphi([L_0, L_m], Y_{n+\frac{1}{2}}) + \varphi([L_m, Y_{n+\frac{1}{2}}], L_0) + \varphi([Y_{n+\frac{1}{2}}, L_0], L_m) = 0$, so

$$(m + n + \frac{1}{2})\varphi(L_m, Y_{n+\frac{1}{2}}) = (n + \frac{1-m}{2})\varphi(L_0, Y_{m+n+\frac{1}{2}}).$$

Then for all $m, n \in \mathbb{Z}$, we get

$$\varphi(L_m, Y_{n+\frac{1}{2}}) = \frac{n + \frac{1-m}{2}}{m + n + \frac{1}{2}}\varphi(L_0, Y_{m+n+\frac{1}{2}}).$$

Consequently, we have

$$\overline{\varphi}(L_m, Y_{n+\frac{1}{2}}) = 0, \qquad m, n \in \mathbb{Z}.$$

From the relation that

$$\varphi([N_0, Y_{m+\frac{1}{2}}], Y_{-m-\frac{1}{2}}) + \varphi([Y_{m+\frac{1}{2}}, Y_{-m-\frac{1}{2}}], N_0) + \varphi([Y_{-m-\frac{1}{2}}, N_0], Y_{m+\frac{1}{2}}) = 0, \quad m \in \mathbb{Z},$$

we have

$$\varphi(Y_{m+\frac{1}{2}}, Y_{-m-\frac{1}{2}}) = (m + \frac{1}{2})\varphi(N_0, M_0), \quad m \in \mathbb{Z}. \tag{3.5}$$

By (3.1), $\varphi([L_m, Y_{p+\frac{1}{2}}], Y_{q+\frac{1}{2}}) + \varphi([Y_{p+\frac{1}{2}}, Y_{q+\frac{1}{2}}], L_m) + \varphi([Y_{q+\frac{1}{2}}, L_m], Y_{p+\frac{1}{2}}) = 0$, then

$$(p+\frac{1-m}{2})\varphi(Y_{m+p+\frac{1}{2}}, Y_{q+\frac{1}{2}}) + (p-q)\varphi(M_{p+q+1}, L_m) - (q+\frac{1-m}{2})\varphi(Y_{m+q+\frac{1}{2}}, Y_{p+\frac{1}{2}}) = 0,$$

for all $m, p, q, \in \mathbb{Z}$. Let $m = -p - q - 1$, then for all $p, q \in \mathbb{Z}$, we have

$$(p+1+\frac{p+q}{2})\varphi(Y_{-q-\frac{1}{2}}, Y_{q+\frac{1}{2}}) + (p-q)\varphi(M_{p+q+1}, L_{-p-q-1}) - (q+1+\frac{p+q}{2})\varphi(Y_{-p-\frac{1}{2}}, Y_{p+\frac{1}{2}}) = 0. \tag{3.6}$$

Using (3.5), we get

$$\varphi(L_{-p-q-1}, M_{p+q+1}) = \frac{p+q+1}{2}\varphi(N_0, M_0), \qquad p \neq q. \tag{3.7}$$

Letting $p = -2, q = 0$ in (3.7), we have

$$\varphi(L_1, M_{-1}) = -\frac{1}{2}\varphi(N_0, M_0) = -\varphi(Y_{\frac{1}{2}}, Y_{-\frac{1}{2}}). \tag{3.8}$$

It follows from (3.5) and (3.7) that

$$\varphi(Y_{m+\frac{1}{2}}, Y_{-m-\frac{1}{2}}) = -(2m + 1)\varphi(L_1, M_{-1}), \quad m \in \mathbb{Z}, \tag{3.9}$$

$$\varphi(L_m, M_{-m}) = -m\varphi(Y_{\frac{1}{2}}, Y_{-\frac{1}{2}}) = m\varphi(L_1, M_{-1}), \qquad m \in \mathbb{Z}. \tag{3.10}$$

By (3.3) and (3.10), we have $\overline{\varphi}(L_m, M_{-m}) = 0$ for all $m \in \mathbb{Z}$. Therefore,

$$\overline{\varphi}(L_m, M_n) = 0, \quad \forall\, m, n \in \mathbb{Z}.$$

According to (3.1), $\varphi([L_m, L_n], N_{-m-n})+\varphi([L_n, N_{-m-n}], L_m)+\varphi([N_{-m-n}, L_m], L_n) = 0$, then we have

$$(n-m)\varphi(L_{m+n}, N_{-m-n}) - (m+n)\varphi(N_{-m}, L_m) + (m+n)\varphi(N_{-n}, L_n) = 0.$$

By (3.4), we get

$$(n-m)c(m+n) + (m+n)c(m) = (m+n)c(n). \tag{3.11}$$

Let $n = 1$ in (3.11) and note $c(0) = c(1) = 0$, then we obtain

$$(m-1)c(m+1) = (m+1)c(m). \tag{3.12}$$

Let $n = 1-m$ in (3.11), then

$$c(m) = c(1-m), \qquad \forall\, m \in \mathbb{Z}.$$

By induction on $m$, we deduce that $c(-1) = c(2)$ determines all $c(m)$ for $m \in \mathbb{Z}$. On the other hand, $c(m) = m^2 - m$ is a solution of equation (3.12). So

$$c(m) = \beta(m^2 - m), \quad \beta \in \mathbb{C},$$

is the general solution of equation (3.12). Therefore,

$$\overline{\varphi}(L_m, N_n) = \delta_{m+n,0}\beta(m^2 - m), \quad \forall\, m, n \in \mathbb{Z},\ \beta \in \mathbb{C}. \tag{3.13}$$

By (3.1), we have

$$\varphi([M_p, Y_{m+\frac{1}{2}}], Y_{n+\frac{1}{2}}) + \varphi([Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}], M_p) + \varphi([Y_{n+\frac{1}{2}}, M_p], Y_{m+\frac{1}{2}}) = 0.$$

Then $(m-n)\varphi(M_{m+n+1}, M_p) = 0$. Let $m+n+1 = q$, then we have $(2m+1-q)\varphi(M_q, M_p) = 0$. This means that

$$\varphi(M_p, M_q) = 0, \quad \forall\, p, q \in \mathbb{Z}.$$

Therefore, we have

$$\overline{\varphi}(M_m, M_n) = \varphi(M_m, M_n) = 0, \quad \forall\, m, n \in \mathbb{Z}. \tag{3.14}$$

By (3.1), $\varphi([L_0, M_m], N_n) + \varphi([M_m, N_n], L_0) + \varphi([N_n, L_0], M_m) = 0$, we have

$$(m+n)\varphi(M_m, N_n) = -2\varphi(L_0, M_{m+n}). \tag{3.15}$$

Then for $m+n \neq 0$, we have

$$\overline{\varphi}(M_m, N_n) = \varphi(M_m, N_n) + \frac{2}{m+n}\varphi(L_0, M_{m+n}) = 0.$$

On the other hand,

$$\overline{\varphi}(M_m, N_{-m}) = \varphi(M_m, N_{-m}) + 2f(M_0) = \varphi(M_m, N_{-m}) - 2\varphi(L_1, M_{-1}).$$

By (3.1), $\varphi([N_p, Y_{m+\frac{1}{2}}], Y_{n+\frac{1}{2}})+\varphi([Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}], N_p)+\varphi([Y_{n+\frac{1}{2}}, N_p], Y_{m+\frac{1}{2}}) = 0$, we have

$$\varphi(Y_{p+m+\frac{1}{2}}, Y_{n+\frac{1}{2}}) + (m-n)\varphi(M_{m+n+1}, N_p) - \varphi(Y_{p+n+\frac{1}{2}}, Y_{m+\frac{1}{2}}) = 0.$$

Let $n = -m-p-1$, then we have

$$\varphi(Y_{p+m+\frac{1}{2}}, Y_{-p-m-\frac{1}{2}}) + (2m+p+1)\varphi(M_{-p}, N_p) - \varphi(Y_{-m-\frac{1}{2}}, Y_{m+\frac{1}{2}}) = 0.$$

By (3.9), we get

$$\varphi(M_{-p},N_p)=2\varphi(L_1,M_{-1}),\quad \forall\, p\in\mathbb{Z}.$$

Therefore,

$$\overline{\varphi}(M_m,N_n)=0,\quad \forall\, m,n\in\mathbb{Z}. \tag{3.16}$$

By (3.1), $\varphi([L_0,M_m],Y_{n+\frac{1}{2}})+\varphi([M_m,Y_{n+\frac{1}{2}}],L_0)+\varphi([Y_{n+\frac{1}{2}},L_0],M_m)=0$, we have

$$(m+n+\frac{1}{2})\varphi(M_m,Y_{n+\frac{1}{2}})=0.$$

So $\varphi(M_m,Y_{n+\frac{1}{2}})=0$ for all $m,n\in\mathbb{Z}$. Therefore

$$\overline{\varphi}(M_m,Y_{n+\frac{1}{2}})=\varphi(M_m,Y_{n+\frac{1}{2}})-f([M_m,Y_{n+\frac{1}{2}}])=0.$$

By (3.1), $\varphi([L_0,N_p],N_q)+\varphi([N_p,N_q],L_0)+\varphi([N_q,L_0],N_p)=0$, we have

$$(p+q)\varphi(N_p,N_q)=0.$$

So $\overline{\varphi}(N_p,N_q)=\varphi(N_p,N_q)=\delta_{p+q,0}k(p)$, where $\overline{\varphi}(N_p,N_{-p})=k(p)$. Then by (3.1), $\varphi([L_{-p-q},N_p],N_q)+\varphi([N_p,N_q],L_{-p-q})+\varphi([N_q,L_{-p-q}],N_p)=0$, we get

$$pk(q)=qk(p).$$

Let $q=1$, then $k(p)=pk(1)$. Set $k(1)=k$, then we have

$$\overline{\varphi}(N_m,N_n)=k\delta_{m+n,0}m,\quad \forall\, m,n\in\mathbb{Z}.$$

By (3.1), $\varphi([L_0,N_m],Y_{n+\frac{1}{2}})+\varphi([N_m,Y_{n+\frac{1}{2}}],L_0)+\varphi([Y_{n+\frac{1}{2}},L_0],N_m)=0$, we have

$$(m+n+\frac{1}{2})\varphi(N_m,Y_{n+\frac{1}{2}})=\varphi(L_0,Y_{m+n+\frac{1}{2}}).$$

Then

$$\begin{aligned}\overline{\varphi}(N_m,Y_{n+\frac{1}{2}}) &= \varphi(N_m,Y_{n+\frac{1}{2}})-f(Y_{m+n+\frac{1}{2}})\\ &= \varphi(N_m,Y_{n+\frac{1}{2}})-\frac{1}{m+n+\frac{1}{2}}\varphi(L_0,Y_{m+n+\frac{1}{2}})\\ &= 0,\quad \forall\, m,n\in\mathbb{Z}.\end{aligned}$$

By (3.1), $\varphi([L_0,Y_{m+\frac{1}{2}}],Y_{n+\frac{1}{2}})+\varphi([Y_{m+\frac{1}{2}},Y_{n+\frac{1}{2}}],L_0)+\varphi([Y_{n+\frac{1}{2}},L_0],Y_{m+\frac{1}{2}})=0$, we have

$$(m+n+1)\varphi(Y_{m+\frac{1}{2}},Y_{n+\frac{1}{2}})=(m-n)\varphi(L_0,M_{m+n+1}). \tag{3.17}$$

For $m+n+1\neq 0$,

$$\begin{aligned}\overline{\varphi}(Y_{m+\frac{1}{2}},Y_{n+\frac{1}{2}}) &= \varphi(Y_{m+\frac{1}{2}},Y_{n+\frac{1}{2}})-f([Y_{m+\frac{1}{2}},Y_{n+\frac{1}{2}}])\\ &= \varphi(Y_{m+\frac{1}{2}},Y_{n+\frac{1}{2}})-(m-n)f(M_{m+n+1})\\ &= \varphi(Y_{m+\frac{1}{2}},Y_{n+\frac{1}{2}})-\frac{m-n}{m+n+1}\varphi(L_0,M_{m+n+1}).\end{aligned}$$

By (3.17), we have

$$\overline{\varphi}(Y_{m+\frac{1}{2}},Y_{n+\frac{1}{2}})=0,\quad m+n+1\neq 0.$$

On the other hand, for all $m \in \mathbb{Z}$,

$$\begin{aligned}\overline{\varphi}(Y_{m+\frac{1}{2}}, Y_{-m-\frac{1}{2}}) &= \varphi(Y_{m+\frac{1}{2}}, Y_{-m-\frac{1}{2}}) - f([Y_{m+\frac{1}{2}}, Y_{-m-\frac{1}{2}}]) \\ &= \varphi(Y_{m+\frac{1}{2}}, Y_{-m-\frac{1}{2}}) - (2m+1)f(M_0) \\ &= \varphi(Y_{m+\frac{1}{2}}, Y_{-m-\frac{1}{2}}) + (2m+1)\varphi(L_1, M_{-1}).\end{aligned}$$

By (3.9), we have

$$\overline{\varphi}(Y_{m+\frac{1}{2}}, Y_{-m-\frac{1}{2}}) = 0, \qquad \forall\, m \in \mathbb{Z}.$$

So

$$\overline{\varphi}(Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}) = 0, \quad \forall\, m, n \in \mathbb{Z}.$$

Therefore, $\overline{\varphi}$ is determined by the following three 2-cocycles

$$\begin{aligned}&\varphi_1(L_m, L_n) = \delta_{m+n,0}\frac{m^3-m}{12}, \\ &\varphi_2(L_m, N_n) = \delta_{m+n,0}(m^2-m), \\ &\varphi_3(N_m, N_n) = \delta_{m+n,0}n.\end{aligned}$$

□

**Remark 3.2.** *It is referred in* [25] *that* $\widetilde{\mathfrak{sv}}$ *has three independent classes of central extensions given by the cocycles*

$$c_1(L_m, L_n) = \delta_{m+n,0}\frac{m^3-m}{12},\ c_2(L_m, N_n) = \delta_{m+n,0}m^2,\ c_3(N_m, N_n) = \delta_{m+n,0}m.$$

*Here we prove in detail that* $\widetilde{\mathfrak{sv}}$ *has only three independent classes of central extensions.*

Let $\mathfrak{g}$ be a perfect Lie algebra, i.e., $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. $(\widehat{\mathfrak{g}}, \pi)$ is called a central extension of $\mathfrak{g}$ if $\pi : \widehat{\mathfrak{g}} \longrightarrow \mathfrak{g}$ is a surjective homomorphism whose kernel lies in the center of the Lie algebra $\widehat{\mathfrak{g}}$. The pair $(\widehat{\mathfrak{g}}, \pi)$ is called a covering of $\mathfrak{g}$ if $\widehat{\mathfrak{g}}$ is perfect. A covering $(\widehat{\mathfrak{g}}, \pi)$ is a universal central extension of $\mathfrak{g}$ if for every central extension $(\widehat{\mathfrak{g}}', \varphi)$ of $\mathfrak{g}$ there is a unique homomorphism $\psi : \widehat{\mathfrak{g}} \longrightarrow \widehat{\mathfrak{g}}'$ for which $\varphi\psi = \pi$. In [6], it is proved that every perfect Lie algebra has a universal central extension.

Let $\widehat{\mathfrak{sv}} = \widetilde{\mathfrak{sv}} \bigoplus \mathbb{C}\ C_L \bigoplus \mathbb{C}\ C_{LN} \bigoplus \mathbb{C}\ C_N$ be a vector space over the complex field $\mathbb{C}$ with a basis $\{L_n, M_n, N_n, Y_{n+\frac{1}{2}}, C_L, C_{LN}, C_N \mid n \in \mathbb{Z}\}$ satisfying the following relations

$$\begin{gathered}[L_m, L_n] = (n-m)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}C_L, \\ [N_m, N_n] = n\delta_{m+n,0}C_N, \\ [L_m, N_n] = nN_{m+n} + \delta_{m+n,0}(m^2-m)C_{LN}, \\ [M_m, M_n] = 0, \quad [Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}] = (m-n)M_{m+n+1}, \\ [L_m, M_n] = nM_{m+n}, \quad [L_m, Y_{n+\frac{1}{2}}] = (n + \frac{1-m}{2})Y_{m+n+\frac{1}{2}}, \\ [N_m, M_n] = 2M_{m+n}, \quad [N_m, Y_{n+\frac{1}{2}}] = Y_{m+n+\frac{1}{2}}, \quad [M_m, Y_{n+\frac{1}{2}}] = 0, \\ [\widehat{\mathfrak{sv}}, C_L] = [\widehat{\mathfrak{sv}}, C_{LN}] = [\widehat{\mathfrak{sv}}, C_N] = 0,\end{gathered}$$

for all $m, n \in \mathbb{Z}$. Denote

$$H = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\, N_n \bigoplus \mathbb{C}\, C_N,\ \mathfrak{Vir} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\, L_n \bigoplus \mathbb{C}\, C_L,\ H_{Vir} = H \bigoplus\ \mathfrak{Vir} \bigoplus \mathbb{C}\, C_{LN},$$

$$\mathcal{S} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\, M_n \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\, Y_{n+\frac{1}{2}}, \quad \mathcal{H}_{\mathcal{S}} = H \bigoplus \mathcal{S}.$$

They are all Lie subalgebras of $\widehat{\mathfrak{sv}}$, where $H$ is an infinite-dimensional Heisenberg algebra, $\mathfrak{Vir}$ is the classical Virasoro algebra, $H_{Vir}$ is the twisted Heisenberg-Virasoro algebra, $\mathcal{S}$ is a two-step nilpotent Lie algebra and $\mathcal{H}_{\mathcal{S}}$ is the semi-direct product of the Heisenberg algebra $H$ and $\mathcal{S}$. Then $\widehat{\mathfrak{sv}}$ is the semi-direct product of the twisted Heisenberg-Virasoro algebra $H_{Vir}$ and $\mathcal{S}$, where $\mathcal{S}$ is an ideal of $\widehat{\mathfrak{sv}}$ .

**Corollary 3.3.** *$\widehat{\mathfrak{sv}}$ is the universal covering algebra of the extended Schrödinger-Virasoro algebra $\widetilde{\mathfrak{sv}}$.* □

Set $deg(C_L) = deg(C_{LN}) = deg(C_N) = 0$. Then there is a $\frac{1}{2}\mathbb{Z}$-grading on $\widehat{\mathfrak{sv}}$ by

$$\widehat{\mathfrak{sv}} = \bigoplus_{n\in\mathbb{Z}} \widehat{\mathfrak{sv}}_{\frac{n}{2}} = (\bigoplus_{n\in\mathbb{Z}} \widehat{\mathfrak{sv}}_n) \bigoplus (\bigoplus_{n\in\mathbb{Z}} \widehat{\mathfrak{sv}}_{n+\frac{1}{2}}),$$

where

$$\begin{aligned} \widehat{\mathfrak{sv}}_n &= span\{L_n, M_n, N_n\},\ \ n \in \mathbb{Z}\setminus\{0\},\\ \widehat{\mathfrak{sv}}_0 &= span\{L_0, M_0, N_0, C_L, C_{LN}, C_N\},\\ \widehat{\mathfrak{sv}}_{n+\frac{1}{2}} &= span\{Y_{n+\frac{1}{2}}\}, \qquad \forall\, n \in \mathbb{Z}. \end{aligned}$$

**Lemma 3.4.** (cf. [1]) *If $\mathfrak{g}$ is a perfect Lie algebra and $\widehat{\mathfrak{g}}$ is a universal central extension of $\mathfrak{g}$, then every derivation of $\mathfrak{g}$ lifts to a derivation of $\widehat{\mathfrak{g}}$. If $\mathfrak{g}$ is centerless, the lift is unique and $Der(\widehat{\mathfrak{g}}) \cong Der(\mathfrak{g})$.* □

It follows from Lemma 3.4 and Corollary 3.3 that

$$D(C_L) = D(C_{LN}) = D(C_N) = 0,$$

for all $D \in \widehat{\mathfrak{sv}}$. Therefore, $Der(\widehat{\mathfrak{sv}}) = Inn(\widehat{\mathfrak{sv}})$.

## 4. The Universal Central Extension of $\widetilde{\mathfrak{sv}}$ in the Category of Leibniz Algebras

The concept of Leibniz algebra was first introduced by Jean-Louis Loday in [16] in his study of the so-called Leibniz homology as a noncommutative analog of Lie algebra homology. A vector space $\mathcal{L}$ equipped with a $\mathbb{C}$-bilinear map $[-,-] : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ is called a Leibniz algebra if the following Leibniz identity satisfies

$$[x,[y,z]] = [[x,y],z] - [[x,z],y], \quad \forall\, x, y, z \in \mathcal{L}. \tag{4.1}$$

Lie algebras are definitely Leibniz algebras. A Leibniz algebra $\mathcal{L}$ is a Lie algebra if and only if $[x,x] = 0$ for all $x \in \mathcal{L}$.

In [16], Jean-Louis Loday and Teimuraz Pirashvili established the concept of universal enveloping algebras of Leibniz algebras and interpreted the Leibniz (co)homology $HL_*$ (resp. $HL^*$) as a Tor-functor (resp. Ext-functor). A bilinear $\mathbb{C}$-valued form $\psi$ on $\mathcal{L}$ is called a Leibniz 2-cocycle if

$$\psi(x,[y,z]) = \psi([x,y],z) - \psi([x,z],y), \quad \forall\, x,y,z \in \mathcal{L}. \tag{4.2}$$

Similar to the 2-cocycle on Lie algebras, a linear function $f$ on $\mathcal{L}$ can induce a Leibniz 2-cocycle $\psi_f$, that is,

$$\psi_f(x,y) = f([x,y]), \qquad \forall\, x,\ y \in \mathcal{L}.$$

Such a Leibniz 2-cocycle is called trival. The one-dimensional Leibniz central extension corresponding to a trivial Leibniz 2-cocycle is also trivial.

In this section, we consider the universal central extension of the extended Schrödinger-Virasoro algebra $\widetilde{\mathfrak{sv}}$ in the category of Leibniz algebras.

Let $\mathfrak{g}$ be a Lie algebra. A bilinear form $f : \mathfrak{g} \longrightarrow \mathbb{C}$ is called invariant if

$$f([x,y],z) = f(x,[y,z]), \quad \forall\, x,y,z \in \mathfrak{g}. \tag{4.3}$$

**Proposition 4.1.** *There is no non-trivial invariant bilinear form on $\widetilde{\mathfrak{sv}}$.*

*Proof.* Let $f : \widetilde{\mathfrak{sv}} \times \widetilde{\mathfrak{sv}} \longrightarrow \mathbb{C}$ be an invariant bilinear form on $\widetilde{\mathfrak{sv}}$.

(1) For $m \neq 0$, we have

$$f(L_m, L_n) = -\frac{1}{m} f([L_m, L_0], L_n) = -\frac{1}{m} f(L_m, [L_0, L_n]) = -\frac{n}{m} f(L_m, L_n),$$

So

$$f(L_m, L_n) = 0, \qquad m+n \neq 0, m \neq 0.$$

Similarly, we have

$$f(L_m, N_n) = 0, \quad f(N_m, N_n) = 0, \qquad m+n \neq 0, m \neq 0.$$

For $n \neq 0$, we have

$$f(L_0, L_n) = \frac{1}{n} f(L_0, [L_0, L_n]) = \frac{1}{n} f([L_0, L_0], L_n) = 0,$$

$$f(L_{-n}, L_n) = \frac{1}{3n} f([L_{-2n}, L_n], L_n) = \frac{1}{3n} f(L_{-2n}, [L_n, L_n]) = 0,$$

Similarly, we can get

$$f(L_0, N_n) = 0,\ f(N_0, N_n) = 0,\ f(L_{-n}, N_n) = 0,\ f(N_{-n}, N_n) = 0.$$

On the other hand,

$$f(L_0, L_0) = \frac{1}{2} f([L_{-1}, L_1], L_0) = \frac{1}{2} f(L_{-1}, [L_1, L_0]) = -\frac{1}{2} f(L_{-1}, L_1) = 0.$$

$$f(L_0, N_0) = \frac{1}{2} f([L_{-1}, L_1], N_0) = \frac{1}{2} f(L_{-1}, [L_1, N_0]) = 0.$$

$$f(N_0, N_0) = f([L_{-1}, N_1], N_0) = f(L_{-1}, [N_1, N_0]) = 0.$$

Therefore,

$$f(L_m, N_n) = 0, \quad f(N_m, N_n) = 0, \quad f(L_m, L_n) = 0, \qquad \forall\, m, n \in \mathbb{Z}.$$

Similarly, we obtain

$$f(L_m, M_n) = 0, \quad f(M_m, M_n) = 0, \quad \forall\, m, n \in \mathbb{Z}.$$

(2) For all $m, n \in \mathbb{Z}$, we have

$$f(L_m, Y_{n+\frac{1}{2}}) = \frac{1}{n+\frac{1}{2}} f(L_m, [L_0, Y_{n+\frac{1}{2}}]) = \frac{1}{n+\frac{1}{2}} f([L_m, L_0], Y_{n+\frac{1}{2}}) = -\frac{m}{n+\frac{1}{2}} f(L_m, Y_{n+\frac{1}{2}}).$$

Then $\dfrac{m+n+\frac{1}{2}}{n+\frac{1}{2}} f(L_m, Y_{n+\frac{1}{2}}) = 0$. Obviously, $f(L_m, Y_{n+\frac{1}{2}}) = 0$ for all $m, n \in \mathbb{Z}$.

(3) For all $m, n \in \mathbb{Z}$, we have

$$f(N_m, M_n) = \frac{1}{2} f(N_m, [N_0, M_n]) = \frac{1}{2} f([N_m, N_0], M_n) = 0,$$

$$f(N_m, Y_{n+\frac{1}{2}}) = f(N_m, [N_0, Y_{n+\frac{1}{2}}]) = f([N_m, N_0], Y_{n+\frac{1}{2}}) = 0,$$

$$f(M_m, Y_{n+\frac{1}{2}}) = \frac{1}{n+\frac{1}{2}} f(M_m, [L_0, Y_{n+\frac{1}{2}}]) = -\frac{1}{n+\frac{1}{2}} f([M_m, Y_{n+\frac{1}{2}}], L_0) = 0,$$

$$\begin{aligned} f(Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}) &= \frac{1}{m+\frac{1}{2}} f([L_0, Y_{m+\frac{1}{2}}], Y_{n+\frac{1}{2}}) = \frac{1}{m+\frac{1}{2}} f(L_0, [Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}]) \\ &= \frac{m-n}{m+\frac{1}{2}} f(L_0, M_{m+n+1}) = 0. \end{aligned}$$

□

**Remark 4.2.** *In fact, it is enough to check that Proposition 4.1 holds for the set of generators* $\{L_{-2}, L_{-1}, L_1, L_2, N_1, Y_{\frac{1}{2}}\}$. *By the proof of Proposition 4.1, we can see that the process of the computation is independent of the symmetry of the bilinear form. Similar to the method in section 4 in* [9], *we can deduce that*

$$HL^2(\widetilde{\mathfrak{sv}}, \mathbb{C}) = H^2(\widetilde{\mathfrak{sv}}, \mathbb{C}),$$

*where* $HL^2(\widetilde{\mathfrak{sv}}, \mathbb{C})$ *is the second Leibniz cohomology group of* $\widetilde{\mathfrak{sv}}$. *That is to say, the universal central extension of* $\widetilde{\mathfrak{sv}}$ *in the category of Leibniz algebras is the same as that in the category of Lie algebras.*

## 5. The Automorphism Group of $\widetilde{\mathfrak{sv}}$

Denote by $Aut(\widetilde{\mathfrak{sv}})$ and $\mathcal{I}$ the automorphism group and the inner automorphism group of $\widetilde{\mathfrak{sv}}$ respectively. Obviously, $\mathcal{I}$ is generated by $\exp(k\mathrm{ad}M_m + l\mathrm{ad}Y_{n+\frac{1}{2}})$, $m, n \in \mathbb{Z}$, $k, l \in \mathbb{C}$.

For convenience, denote

$$L = span\{L_n \mid n \in \mathbb{Z}\}, \quad N = span\{N_n \mid n \in \mathbb{Z}\},$$

$$M = span\{M_n \mid n \in \mathbb{Z}\}, \quad Y = span\{Y_{n+\frac{1}{2}} \mid n \in \mathbb{Z}\}.$$

**Lemma 5.1.** *Let $\sigma \in Aut(\widetilde{\mathfrak{sv}})$, then*

$$\sigma(M_n) \in M, \quad \sigma(Y_{n+\frac{1}{2}}) \in M + Y, \quad \sigma(N_n) \in M + Y + N,$$

*for all $n \in \mathbb{Z}$. In particular,*

$$\sigma(N_0) = \sum_{i=p}^{q} a_i M_i + N_0 + \sum_{j=s}^{t} b_j Y_{j+\frac{1}{2}},$$

*for some $a_i, b_j \in \mathbb{C}$ and $p, q, s, t \in \mathbb{Z}$.*

*Proof.* Let $I$ be a nontrivial ideal of $\widetilde{\mathfrak{sv}}$. Then $I$ is a $L_0$-module. Since the decomposition of eigenvalue subspace of $L_0$ is in concordance with the $\frac{1}{2}\mathbb{Z}$-grading of $\widetilde{\mathfrak{sv}}$, we have

$$I = \bigoplus_{n\in\mathbb{Z}} I_{\frac{n}{2}} = \bigoplus_{n\in\mathbb{Z}} I \bigcap \widetilde{\mathfrak{sv}}_{\frac{n}{2}}.$$

Hence, there exists some $n \in \mathbb{Z}$ such that $aL_n + bM_n + cN_n \in I$ or $Y_{n+\frac{1}{2}} \in I$, where $a, b, c \in \mathbb{C}$ and not all zero. If $aL_n + bM_n + cN_n \in I$, then

$$[aL_n + bM_n + cN_n, M_0] = 2cM_n \in I, \ \ [aL_n + bM_n + cN_n, N_0] = -2bM_n \in I.$$

If $b = c = 0$, then $a \neq 0$ and $L_n \in$ I. But $[L_n, \widetilde{\mathfrak{sv}}] = \widetilde{\mathfrak{sv}}$ for any $n \in \mathbb{Z}$, then we have $I = \widetilde{\mathfrak{sv}}$, a contradiction. So $b \neq 0$ or $c \neq 0$. Therefore, $M_n \in I$, and we have $aL_n + cN_n \in I$. Since $[aL_n + cN_n, N_1] = aN_{n+1} \in I$, we get $N_{n+1} \in I$ if $a \neq 0$.

(1) If there exists some $M_n \in I$, by the fact that $[N_{m-n}, M_n] = 2M_m$ for all $m \in \mathbb{Z}$, we obtain $M \subseteq I$.

(2) If there exists some $N_n \in I$ and $n \neq 0$, then $N \subseteq I$ since $[L_{m-n}, N_n] = nN_m \in I$ for all $m \in \mathbb{Z}$. On the other hand, we have

$$[N_0, M_m] = 2M_m, \ \ [N_0, Y_{m+\frac{1}{2}}] = Y_{m+\frac{1}{2}},$$

for all $m \in \mathbb{Z}$. So $M \subseteq I$, $Y \subseteq I$, and therefore $N \bigoplus M \bigoplus Y \subseteq I$.

If $N_0 \in I$, according to the proof above, we have $M \subseteq I, Y \subseteq I$. In addition, $[L, N_0] = 0$ and $[N, N_0] = 0$, so $\mathbb{C}N_0 \bigoplus M \bigoplus Y \subseteq I$.

(3) If there exists some $Y_{n+\frac{1}{2}} \in I$, we have $Y \subseteq I$ since $[N_{m-n}, Y_{n+\frac{1}{2}}] = Y_{m+\frac{1}{2}}$ for all $m \in \mathbb{Z}$. Moreover, $[Y_{m+\frac{1}{2}}, Y_{\frac{1}{2}}] = mM_{m+1}$ for all $m \in \mathbb{Z}$ and $[Y_{1+\frac{1}{2}}, Y_{-1+\frac{1}{2}}] = 2M_1$, so $M \subseteq I$.

Set $\mathfrak{I}_1 = M$, $\mathfrak{I}_2 = M \bigoplus Y$, $\mathfrak{I}_3 = M \bigoplus \mathbb{C}N_0 \bigoplus Y$, $\mathfrak{I}_4 = M \bigoplus N \bigoplus Y$. Then $I = \mathfrak{I}_k$ for some $k = 1, 2, 3, 4$. Obviously, $\mathfrak{I}_1$ and $\mathfrak{I}_2$ both have infinite-dimensional center $M$, while the center of $\mathfrak{I}_3$ and $\mathfrak{I}_4$ are zero, i.e.,

$$C(\mathfrak{I}_1) = C(\mathfrak{I}_2) = M, \quad C(\mathfrak{I}_3) = C(\mathfrak{I}_4) = 0.$$

For any $\sigma \in Aut(\widetilde{\mathfrak{sv}})$, $\sigma(I)$ is still a non-trivial ideal of $\widetilde{\mathfrak{sv}}$ and $\sigma(C(I)) = C(\sigma(I))$. Then

$$\sigma(\mathfrak{I}_\mathfrak{i}) = \mathfrak{I}_\mathfrak{j}, \ \ i, j = 1, 2; \quad \sigma(\mathfrak{I}_k) = \mathfrak{I}_l, \ \ k, l = 3, 4.$$

If $\sigma(\mathfrak{I}_1)=\mathfrak{I}_2$, then for every $m\in\mathbb{Z}$, there exists unique $x_m=\sum a_{m_i}M_{m_i}\in\mathfrak{I}_1$ such that $\sigma(x_m)=Y_{m+\frac{1}{2}}$. Then $(m-n)M_{m+n+1}=0$ for all $m,n\in\mathbb{Z}$, which is impossible. Therefore,

$$\sigma(\mathfrak{I}_i)=\mathfrak{I}_i,\quad i=1,2.$$

Moreover, we obtain

$$\sigma(M_n)\in M,\quad \sigma(Y_{n+\frac{1}{2}})\in M+Y. \tag{5.1}$$

Assume that

$$\sigma(N_0)=\sum a_iM_i+\sum b_jN_j+\sum c_kY_{k+\frac{1}{2}},$$

where $a_i,b_j,c_k\in\mathbb{C}$. According to (5.1), $\sigma(M_0)\in M$. So there exist some $f(m)\in\mathbb{C}^*$ such that $\sigma(M_0)=\sum f(m)M_m$. By $\sigma[N_0,M_0]=[\sigma(N_0),\sigma(M_0)]$, we get

$$\sum_m f(m)M_m=\sum_{m,j} b_jf(m)M_{m+j}. \tag{5.2}$$

Set $p=min\{m\in\mathbb{Z}|f(m)\neq 0\}$, $q=max\{m\in\mathbb{Z}|f(m)\neq 0\}$. If $j\neq 0$, we have

$$p+j<p\ \text{ if } j<0;\quad (\text{resp. } q+j>q\ \text{ if } j>0).$$

By (5.2), it is easy to see that $b_jf(p)=0$ ( resp. $b_jf(q)=0$). So $b_j=0$ for all $j\neq 0$. Then by (5.2), $b_0=1$. Therefore,

$$\sigma(N_0)=\sum a_iM_i+N_0+\sum c_kY_{k+\frac{1}{2}}.$$

This forces that $\sigma(\mathfrak{I}_k)=\mathfrak{I}_k,\ k=3,4.$ $\square$

**Lemma 5.2.** *For any $\sigma\in Aut(\widetilde{\mathfrak{sv}})$, there exist some $\pi\in\mathcal{I}$ and $\epsilon\in\{\pm1\}$ such that*

$$\bar{\sigma}(L_n)=a^n\epsilon L_{\epsilon n}+a^n\lambda N_{\epsilon n}, \tag{5.3}$$

$$\bar{\sigma}(N_n)=a^nN_{\epsilon n}, \tag{5.4}$$

$$\bar{\sigma}(M_n)=\epsilon d^2a^{n-1}M_{\epsilon(n-2\lambda)}, \tag{5.5}$$

$$\bar{\sigma}(Y_{n+\frac{1}{2}})=da^nY_{\epsilon(n+\frac{1}{2}-\lambda)}, \tag{5.6}$$

*where $\bar{\sigma}=\pi^{-1}\sigma,\lambda\in\mathbb{Z}$ and $a,d\in\mathbb{C}^*$. Conversely, if $\bar{\sigma}$ is a linear operator on $\widetilde{\mathfrak{sv}}$ satisfying (5.3)-(5.6) for some $\epsilon\in\{\pm1\}$, $\lambda\in\mathbb{Z}$ and $a,d\in\mathbb{C}^*$, then $\bar{\sigma}\in Aut(\widetilde{\mathfrak{sv}})$.*

*Proof.* By Lemma 5.1, for all $\sigma\in Aut(\widetilde{\mathfrak{sv}})$, $\sigma(N_0)=\sum\limits_{i=p}^{q}a_iM_i+N_0+\sum\limits_{j=s}^{t}b_jY_{j+\frac{1}{2}}$ for some $a_i,b_j\in\mathbb{C}$ and $p,q,s,t\in\mathbb{Z}$. Let

$$\pi=\prod_{j=s}^{t}exp(-b_j\text{ad}Y_{j+\frac{1}{2}})\prod_{i=p}^{q}exp(-\frac{a_i}{2}\text{ad}M_i)\prod_{i,j=s}^{t}exp(\frac{i-j}{4}b_ib_j\text{ad}M_{i+j+1})\in\mathcal{I},$$

then we can deduce that $\sigma(N_0)=\pi(N_0)$, that is,

$$\pi^{-1}\sigma(N_0)=N_0.$$

Set $\bar{\sigma} = \pi^{-1}\sigma$. By $[N_0, \bar{\sigma}(L_m)] = [N_0, \bar{\sigma}(N_m)] = 0$ and $[N_0, \bar{\sigma}(Y_{m+\frac{1}{2}})] = \bar{\sigma}(Y_{m+\frac{1}{2}})$ for all $m \in \mathbb{Z}$, we get

$$\bar{\sigma}(L_m) \in L + N, \ \ \bar{\sigma}(N_m) \in N, \ \ \bar{\sigma}(Y_{m+\frac{1}{2}}) \in Y.$$

For any $\bar{\sigma} \in Aut(\widetilde{\mathfrak{sv}})$, denote $\bar{\sigma}|_L = \bar{\sigma}'$. By the automorphisms of the classical Witt algebra, $\bar{\sigma}'(L_m) = \epsilon a^m L_{\epsilon m}$ for all $m \in \mathbb{Z}$, where $a \in \mathbb{C}^*$ and $\epsilon \in \{\pm 1\}$. Assume that

$$\bar{\sigma}(L_0) = \epsilon L_0 + \sum \lambda_i N_i, \quad \bar{\sigma}(L_n) = a^n \epsilon L_{\epsilon n} + a^n \sum \lambda(n_i) N_{n_i}, \ \ n \neq 0,$$
$$\bar{\sigma}(N_n) = a^n \sum \mu(n_j) N_{n_j}, \bar{\sigma}(M_n) = a^n \sum f(n_r) M_{n_r}, \bar{\sigma}(Y_{n+\frac{1}{2}}) = a^n \sum h(n_t + \frac{1}{2}) Y_{n_t+\frac{1}{2}},$$

where each formula is of finite terms and $\mu(n_j), f(n_r), h(n_t + \frac{1}{2}) \in \mathbb{C}^*$, $\lambda(i), \lambda(n_i) \in \mathbb{C}$. From $[\bar{\sigma}(L_0), \bar{\sigma}(M_m)] = m\bar{\sigma}(M_m)$, we have

$$\sum \epsilon m_r f(m_r) M_{m_r} + 2 \sum \lambda_i f(m_r) M_{i+m_r} = m \sum f(m_r) M_{m_r}.$$

This forces that $\lambda_i = 0$ for $i \neq 0$ and $\epsilon m_r + 2\lambda_0 = m$. So $m_r = \epsilon(m - 2\lambda_0)$ and

$$\bar{\sigma}(L_0) = \epsilon L_0 + \lambda_0 N_0, \quad \bar{\sigma}(M_n) = a^n f(\epsilon(n - 2\lambda_0)) M_{\epsilon(n-2\lambda_0)},$$

for all $n \in \mathbb{Z}$. From $[\bar{\sigma}(L_n), \bar{\sigma}(M_0)] = 0$, we get

$$\lambda_0 M_{\epsilon n - 2\epsilon\lambda_0} = \sum \lambda_{n_i} M_{n_i - 2\epsilon\lambda_0}.$$

Then $n_i = \epsilon n$ and $\lambda_{\epsilon n} = \lambda_0$ for all $n \in \mathbb{Z}$. Therefore,

$$\bar{\sigma}(L_n) = a^n \epsilon L_{\epsilon n} + a^n \lambda_0 N_{\epsilon n}, \quad \text{for all n} \in \mathbb{Z}.$$

Since $[\bar{\sigma}(L_0), \bar{\sigma}(N_n)] = n\bar{\sigma}(N_n)$, we have $\sum(\epsilon n_j - n)\mu(n_j)N_{n_j} = 0$. Obviously, $n_j = \epsilon n$ and

$$\bar{\sigma}(N_n) = a^n \mu(\epsilon n) N_{\epsilon n},$$

for all $n \in \mathbb{Z}$, where $\mu(0) = 1$. Comparing the coefficients of $Y_{n_t+\frac{1}{2}}$ on the both sides of $[\bar{\sigma}(L_0), \bar{\sigma}(Y_{n+\frac{1}{2}})] = (n + \frac{1}{2})\bar{\sigma}(Y_{n+\frac{1}{2}})$, we obtain $n_t + \frac{1}{2} = \epsilon(n + \frac{1}{2} - \lambda_0)$, which implies that $\lambda_0 \in \mathbb{Z}$. So

$$\bar{\sigma}(Y_{n+\frac{1}{2}}) = a^n h(\epsilon(n + \frac{1}{2} - \lambda_0)) Y_{\epsilon(n+\frac{1}{2}-\lambda_0)}, \quad \text{for all n} \in \mathbb{Z}.$$

By $[\bar{\sigma}(N_n), \bar{\sigma}(M_m)] = 2\bar{\sigma}(M_{m+n})$, we get

$$\mu(\epsilon n) f(\epsilon(m - 2\lambda_0)) = f(\epsilon(m + n - 2\lambda_0)).$$

Letting $m = 2\lambda_0$, we obtain

$$f(\epsilon n) = f(0)\mu(\epsilon n).$$

By the coefficients of $Y_{\epsilon(m+n+\frac{1}{2}-\lambda_0)}$ on the both sides of $[\bar{\sigma}(N_m), \bar{\sigma}(Y_{n+\frac{1}{2}})] = \bar{\sigma}(Y_{m+n+\frac{1}{2}})$, we have

$$h(\epsilon(m + \frac{1}{2})) = \mu(\epsilon m) h(\frac{\epsilon}{2}).$$

Similarly, comparing the coefficients of $N_{\epsilon(m+n)}$ on the both sides of $[\bar{\sigma}(L_n), \bar{\sigma}(N_m)] = m\bar{\sigma}(N_{m+n})$, we have $m\mu(\epsilon m) = m\mu(\epsilon(m+n))$ for all $m, n \in \mathbb{Z}$. Then

$$\mu(\epsilon m) = \mu(0) = 1,$$

for all $m \in \mathbb{Z}$. Therefore,

$$f(\epsilon m) = f(0), \quad h(\epsilon(m+\frac{1}{2})) = h(\frac{\epsilon}{2}).$$

Finally, we deduce that $\epsilon h(\frac{\epsilon}{2})^2 = af(0)$ by comparing the coefficient of $M_{\epsilon(m+n+1-2\lambda_0)}$ on the both sides of $[\bar{\sigma}(Y_{m+\frac{1}{2}}), \bar{\sigma}(Y_{n+\frac{1}{2}})] = (m-n)\bar{\sigma}(M_{m+n+1})$. Let $d = h(\frac{\epsilon}{2})$, then $f(0) = \epsilon a^{-1} d^2$. Therefore,

$$\begin{aligned}
&\bar{\sigma}(L_n) = a^n \epsilon L_{\epsilon n} + a^n \lambda_0 N_{\epsilon n}, \quad \bar{\sigma}(N_n) = a^n N_{\epsilon n}, \\
&\bar{\sigma}(M_n) = \epsilon d^2 a^{n-1} M_{\epsilon(n-2\lambda_0)}, \quad \bar{\sigma}(Y_{n+\frac{1}{2}}) = d a^n Y_{\epsilon(n+\frac{1}{2}-\lambda_0)}.
\end{aligned}$$

It is easy to check the converse part of the theorem.

□

Denote by $\bar{\sigma}(\epsilon, \lambda, a, d)$ the automorphism of $\widetilde{\mathfrak{sv}}$ satisfying (5.3)-(5.6), then

$$\bar{\sigma}(\epsilon_1, \lambda_1, a_1, d_1)\bar{\sigma}(\epsilon_2, \lambda_2, a_2, d_2) = \bar{\sigma}(\epsilon_1\epsilon_2, \epsilon_2\lambda_1 + \lambda_2, a_1^{\epsilon_2} a_2, d_1 d_2 a_1^{\frac{\epsilon_2 - 1}{2} - \epsilon_2\lambda_2}), \tag{5.7}$$

and $\bar{\sigma}(\epsilon_1, \lambda_1, a_1, d_1) = \bar{\sigma}(\epsilon_2, \lambda_2, a_2, d_2)$ if and only if $\epsilon_1 = \epsilon_2, \lambda_1 = \lambda_2, a_1 = a_2, d_1 = d_2$. Let

$$\bar{\pi}_\epsilon = \bar{\sigma}(\epsilon, 0, 1, 1), \quad \bar{\sigma}_\lambda = (1, \lambda, 1, 1), \quad \bar{\sigma}_{a,d} = (1, 0, a, d)$$

and

$$\mathfrak{a} = \{\bar{\pi}_\epsilon \mid \epsilon = \pm 1\}, \quad \mathfrak{t} = \{\bar{\sigma}_\lambda \mid \lambda \in \mathbb{Z}\}, \quad \mathfrak{b} = \{\bar{\sigma}_{a,d} \mid a, d \in \mathbb{C}^*\}.$$

By (5.7), we have the following relations:

$$\bar{\sigma}(\epsilon, \lambda, a, d) = \bar{\sigma}(\epsilon, 0, 1, 1)\bar{\sigma}(1, \lambda, 1, 1)\bar{\sigma}(1, 0, a, d) \in \mathfrak{atb},$$

$$\bar{\sigma}(\epsilon, \lambda, a, d)^{-1} = \bar{\sigma}(\epsilon, -\epsilon\lambda, a^{-\epsilon}, d^{-1} a^{\frac{1-\epsilon}{2} - \lambda}),$$

$$\bar{\pi}_{\epsilon_1}\bar{\pi}_{\epsilon_2} = \bar{\pi}_{\epsilon_1\epsilon_2}, \quad \bar{\sigma}_{\lambda_1}\bar{\sigma}_{\lambda_2} = \bar{\sigma}_{\lambda_1+\lambda_2}, \quad \bar{\sigma}_{a_1,d_1}\bar{\sigma}_{a_2,d_2} = \bar{\sigma}_{a_1a_2,d_1d_2},$$

$$\bar{\pi}_\epsilon^{-1}\bar{\sigma}_\lambda\bar{\pi}_\epsilon = \bar{\sigma}_{\epsilon\lambda}, \quad \bar{\pi}_\epsilon^{-1}\bar{\sigma}_{a,d}\bar{\pi}_\epsilon = \bar{\sigma}_{a^\epsilon, da^{\frac{\epsilon-1}{2}}}, \quad \bar{\sigma}_\lambda^{-1}\bar{\sigma}_{a,d}\bar{\sigma}_\lambda = \bar{\sigma}_{a,da^{-\lambda}}.$$

Hence, the following lemma holds.

**Lemma 5.3.** $\mathfrak{a}, \mathfrak{t}$ *and* $\mathfrak{b}$ *are all subgroups of* $Aut(\widetilde{\mathfrak{sv}})$ *and*

$$Aut(\widetilde{\mathfrak{sv}}) = \mathcal{I} \rtimes ((\mathfrak{a} \ltimes \mathfrak{t}) \ltimes \mathfrak{b}),$$

*where* $\mathfrak{a} \cong \mathbb{Z}_2 = \{\pm 1\}, \mathfrak{t} \cong \mathbb{Z}$, $\mathfrak{b} \cong \mathbb{C}^* \times \mathbb{C}^*$.

□

Let $\mathbb{C}^\infty = \{(a_i)_{i\in\mathbb{Z}} \mid a_i \in \mathbb{C},$ all but a finite of the $a_i$ are zero $\}$, $\mathcal{I}_\mathcal{C}$ a subgroup of $\mathcal{I}$ generated by $\{\exp(k\mathrm{ad}M_n) \mid n \in \mathbb{Z}, k \in \mathbb{C}\}$ and $\overline{\mathcal{I}} = \mathcal{I}/\mathcal{I}_\mathcal{C}$ the quotient group of $\mathcal{I}$. Then $\mathbb{C}^\infty$ is an abelian group and $\mathcal{I}_\mathcal{C}$ is an abelian normal subgroup of $\mathcal{I}$. As a matter of fact, $\mathcal{I}_\mathcal{C}$ is the center of the group $\mathcal{I}$.

Note $(\mathrm{ad}M_i)^2 = (\mathrm{ad}Y_{j+\frac{1}{2}})^3 = \mathrm{ad}M_i\mathrm{ad}Y_{j+\frac{1}{2}} = \mathrm{ad}Y_{j+\frac{1}{2}}\mathrm{ad}M_i = 0$ for all $i, j \in \mathbb{Z}$, then

$$exp(\alpha\mathrm{ad}M_i) = 1 + \alpha\mathrm{ad}M_i,$$

$$exp(\beta\mathrm{ad}Y_{j+\frac{1}{2}}) = 1 + \beta\mathrm{ad}Y_{j+\frac{1}{2}} + \frac{1}{2}\beta^2(\mathrm{ad}Y_{j+\frac{1}{2}})^2,$$

$$exp(\alpha\mathrm{ad}M_i)exp(\beta\mathrm{ad}Y_{j+\frac{1}{2}}) = exp(\beta\mathrm{ad}Y_{j+\frac{1}{2}}) + \alpha\mathrm{ad}M_i,$$

for all $\alpha, \beta \in \mathbb{C}$. Furthermore, we get

$$\begin{aligned}
& exp(b_{m_1}\mathrm{ad}Y_{m_1+\frac{1}{2}})exp(b_{m_2}\mathrm{ad}Y_{m_2+\frac{1}{2}})\cdots exp(b_{m_t}\mathrm{ad}Y_{m_t+\frac{1}{2}}) \\
= \; & 1 + \sum_{k=1}^{t} b_{m_k}\mathrm{ad}Y_{m_k+\frac{1}{2}} + \sum_{k=1}^{t}\frac{b_{m_k}^2}{2}(\mathrm{ad}Y_{m_k+\frac{1}{2}})^2 + \sum_{1\le i<j\le t} b_{m_i}b_{m_j}\mathrm{ad}Y_{m_i+\frac{1}{2}}\mathrm{ad}Y_{m_j+\frac{1}{2}} \\
= \; & exp(\sum_{k=1}^{t} b_{m_k}\mathrm{ad}Y_{m_k+\frac{1}{2}}) + \frac{1}{2}\sum_{1\le i<j\le t} b_{m_i}b_{m_j}(\mathrm{ad}Y_{m_i+\frac{1}{2}}\mathrm{ad}Y_{m_j+\frac{1}{2}} - \mathrm{ad}Y_{m_j+\frac{1}{2}}\mathrm{ad}Y_{m_i+\frac{1}{2}}) \\
= \; & exp(\sum_{k=1}^{t} b_{m_k}\mathrm{ad}Y_{m_k+\frac{1}{2}}) + \sum_{1\le i<j\le t}\frac{m_i - m_j}{2}b_{m_i}b_{m_j}\mathrm{ad}M_{m_i+m_j+1} \\
= \; & exp(\sum_{k=1}^{t} b_{m_k}\mathrm{ad}Y_{m_k+\frac{1}{2}})exp(\sum_{1\le i<j\le t}\frac{m_i - m_j}{2}b_{m_i}b_{m_j}\mathrm{ad}M_{m_i+m_j+1}),
\end{aligned}$$

for all $m_k \in \mathbb{Z}, b_{m_k} \in \mathbb{C}, 1 \le k \le t$. Therefore,

$$exp(b_{m_1}\mathrm{ad}Y_{m_1+\frac{1}{2}})exp(b_{m_2}\mathrm{ad}Y_{m_2+\frac{1}{2}})\cdots exp(b_{m_t}\mathrm{ad}Y_{m_t+\frac{1}{2}})\mathcal{I}_\mathcal{C} = exp(\sum_{k=1}^{t} b_{m_k}\mathrm{ad}Y_{m_k+\frac{1}{2}})\mathcal{I}_\mathcal{C}. \tag{5.8}$$

**Lemma 5.4.** *$\mathcal{I}_\mathcal{C}$ and $\overline{\mathcal{I}}$ are isomorphic to $\mathbb{C}^\infty$.*

*Proof.* Define $f : \mathcal{I}_\mathcal{C} \longrightarrow \mathbb{C}^\infty$ by

$$f(\prod_{i=1}^{s} exp(\alpha_{k_i}\mathrm{ad}M_{k_i})) = (a_p)_{p\in\mathbb{Z}},$$

where $a_{k_i} = \alpha_{k_i}$ for $1 \le i \le s$, and the others are zero, $k_i \in \mathbb{Z}$ and $k_1 < k_2 < \cdots < k_s$. Since every element of $\mathcal{I}_\mathcal{C}$ has the unique form of $\prod_{i=1}^{s} exp(\alpha_{k_i}\mathrm{ad}M_{k_i})$, it is easy to check that $f$ is an isomorphism of group.

Similar to the proof above, we have $\overline{\mathcal{I}} \cong \mathbb{C}^\infty$ via (5.8). $\square$

**Theorem 5.5.** *$Aut(\widetilde{\mathfrak{sv}}) = (\mathcal{I}_{\mathcal{C}} \rtimes \overline{\mathcal{I}}) \rtimes ((\mathfrak{a} \ltimes \mathfrak{t}) \ltimes \mathfrak{b}) \cong (\mathbb{C}^\infty \rtimes \mathbb{C}^\infty) \rtimes ((\mathbb{Z}_2 \ltimes \mathbb{Z}) \ltimes (\mathbb{C}^* \times \mathbb{C}^*))$.* □

**Lemma 5.6.** (cf. [21]) *Let $\mathfrak{g}$ be a perfect Lie algebra and let $\widehat{\mathfrak{g}}$ be its universal covering algebra of $\mathfrak{g}$. Then every automorphism $\sigma$ of $\mathfrak{g}$ admits a unique extension $\widetilde{\sigma}$ to an automorphism of $\widehat{\mathfrak{g}}$. Furthermore, the map $\sigma \mapsto \widetilde{\sigma}$ is a group monomorphism.*

We will use Lemma 5.6 to obtain all the automorphisms of $\widehat{\mathfrak{sv}}$ from those of $\widetilde{\mathfrak{sv}}$.

For a perfect Lie algebra $\mathfrak{g}$, its universal covering algebra is constucted as follows in [21]. Let $V = \Lambda^2\mathfrak{g}/J$, where

$$J = span\{x \wedge [y, z] + y \wedge [z, x] + z \wedge [x, y] \mid x, y, z \in \mathfrak{g}\}$$

is a subspace of $\Lambda^2\mathfrak{g}$. Then there is a natural Lie algebra structure in the space $\widetilde{\mathfrak{g}} = \mathfrak{g} \bigoplus V$ with the following bracket

$$[x + u, y + v] = [x, y] + x \vee y,$$

for all $x, y \in \mathfrak{g}, u, v \in V$, where $x \vee y$ is the image of $x \wedge y$ in $V$ under the canonical morphism $\Lambda^2\mathfrak{g} \longrightarrow V$. Then the derived algebra $\widehat{\mathfrak{g}} = [\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}]$ of $\widetilde{\mathfrak{g}}$ is the universal central extension of $\mathfrak{g}$. In face, given $x \in \mathfrak{g}$ there exists $c \in V$ such that $x + c \in \widehat{\mathfrak{g}}$. Then the canonical map $\widehat{\mathfrak{g}} \longrightarrow \mathfrak{g}$ is onto with kernel $\mathfrak{c} \subset V$ and the resulting central extension

$$\{0\} \longrightarrow \mathfrak{c} \longrightarrow \widehat{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow \{0\}$$

of $\mathfrak{g}$ is universal in the sense that there exists a unique morphism from it into any other given central extension of $\mathfrak{g}$.

For any $\theta \in Aut(\mathfrak{g})$, $\theta$ induces an automorphism $\theta_V$ of $V$ via

$$\theta_V(x \vee y) = \theta(x) \vee \theta(y),$$

for all $x, y \in \mathfrak{g}$. Obviously, $\theta$ extends to an automorphism $\theta_{\mathfrak{e}}$ of $\widetilde{\mathfrak{g}}$ by

$$\theta_{\mathfrak{e}}(x + v) = \theta(x) + \theta_V(v),$$

for all $x \in \mathfrak{g}, v \in V$. By restriction, $\theta_{\mathfrak{e}}$ induces an automorphism $\tilde{\theta}$ of $\widehat{\mathfrak{g}}$.

In the following section, we will describe the automorphism group of the universal central extension of $\widehat{\mathfrak{sv}}$ using the above method. Firstly, we have the following lemmas.

**Lemma 5.7.** *In $V = \Lambda^2(\widetilde{\mathfrak{sv}})/J$, we have the following relations for all $m, n \in \mathbb{Z}$ :*

$$L_m \vee L_n = \frac{n-m}{m+n} L_0 \vee L_{m+n}, \quad m+n \neq 0;$$
$$L_m \vee L_{-m} = \frac{m^3-m}{6} L_2 \vee L_{-2};$$
$$L_m \vee N_n = \frac{n}{m+n} L_0 \vee N_{m+n}, \quad m+n \neq 0;$$
$$L_m \vee N_{-m} = \frac{m^2+m}{2}(L_1 \vee N_{-1} + L_{-1} \vee N_1) - mL_{-1} \vee N_1;$$
$$L_m \vee M_n = \frac{n}{2} N_0 \vee M_{m+n}; \quad L_m \vee Y_{n+\frac{1}{2}} = (n + \frac{1-m}{2}) N_0 \vee Y_{m+n+\frac{1}{2}};$$
$$N_m \vee N_n = m\delta_{m+n,0} N_1 \vee N_{-1}; \; N_m \vee M_n = N_0 \vee M_{m+n}; \; N_m \vee Y_{n+\frac{1}{2}} = N_0 \vee Y_{m+n+\frac{1}{2}};$$
$$M_m \vee M_n = M_m \vee Y_{n+\frac{1}{2}} = 0; \quad Y_{m+\frac{1}{2}} \vee Y_{n+\frac{1}{2}} = \frac{m-n}{2} N_0 \vee M_{m+n+1}.$$

□

Using Lemma 5.7, we have the following result.

**Lemma 5.8.** *The universal central extension of $\widetilde{\mathfrak{sv}}$, denoted by $\widehat{\mathfrak{sv}}$, has a basis $\{L'_n, M'_n, N'_n, Y'_{n+\frac{1}{2}}, C_L, C_{LN}, C_N \mid n \in \mathbb{Z}\}$ with the following products:*

$$[L'_m, L'_n] = (n-m)L'_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12} C_L,$$
$$[N'_m, N'_n] = n\delta_{m+n,0} C_N,$$
$$[L'_m, N'_n] = nN'_{m+n} + \delta_{m+n,0}(n^2-n) C_{LN},$$
$$[M'_m, M'_n] = 0, \quad [Y'_{m+\frac{1}{2}}, Y'_{n+\frac{1}{2}}] = (m-n) M'_{m+n+1},$$
$$[L'_m, M'_n] = nM'_{m+n}, \quad [L'_m, Y'_{n+\frac{1}{2}}] = (n + \frac{1-m}{2}) Y'_{m+n+\frac{1}{2}},$$
$$[N'_m, M'_n] = 2M'_{m+n}, \quad [N'_m, Y'_{n+\frac{1}{2}}] = Y'_{m+n+\frac{1}{2}}, \quad [M'_m, Y'_{n+\frac{1}{2}}] = 0,$$
$$[\widehat{\mathfrak{sv}}, C_L] = [\widehat{\mathfrak{sv}}, C_{LN}] = [\widehat{\mathfrak{sv}}, C_N] = 0,$$

*where*

$$L'_0 = L_0, \quad N'_0 = N_0 + L_{-1} \vee N_1;$$
$$L'_m = L_m + \frac{1}{m} L_0 \vee L_m, \quad N'_m = N_m + \frac{1}{m} L_0 \vee N_m, \quad m \neq 0;$$
$$M'_n = M_n + \frac{1}{2} N_0 \vee M_n, \quad Y'_{n+\frac{1}{2}} = Y_{n+\frac{1}{2}} + N_0 \vee Y_{n+\frac{1}{2}}, \quad n \in \mathbb{Z};$$
$$C_L = 2L_2 \vee L_{-2}, \quad C_{LN} = \frac{1}{2}(L_1 \vee N_{-1} + L_{-1} \vee N_1), \quad C_N = N_{-1} \vee N_1.$$

□

**Lemma 5.9.** *For any $\tilde{\theta} \in Aut(\widehat{\mathfrak{sv}})/\mathcal{I}$, we have*

$$\tilde{\theta}(L'_n) = a^n \epsilon L'_{\epsilon n} + a^n \lambda N'_{\epsilon n} - \lambda \delta_{n,0} C_{LN} + \frac{\epsilon}{2} \lambda^2 \delta_{n,0} C_N, \tag{5.9}$$

$$\tilde{\theta}(N'_n) = a^n N'_{\epsilon n} + (\epsilon - 1)\delta_{n,0} C_{LN} + \epsilon \lambda \delta_{n,0} C_N, \tag{5.10}$$

$$\tilde{\theta}(M'_n) = \epsilon d^2 a^{n-1} M'_{\epsilon(n-2\lambda)}, \tag{5.11}$$

$$\tilde{\theta}(Y'_{n+\frac{1}{2}}) = d a^n Y'_{\epsilon(n+\frac{1}{2}-\lambda)}, \tag{5.12}$$

$$\tilde{\theta}(C_L) = \epsilon C_L,\ \tilde{\theta}(C_{LN}) = \epsilon C_{LN},\ \tilde{\theta}(C_N) = \epsilon C_N, \tag{5.13}$$

*for all $n \in \mathbb{Z}$, where $\epsilon \in \{\pm 1\}$, $\lambda \in \mathbb{Z}$, $a, d \in \mathbb{C}^*$. Conversely, if $\tilde{\theta}$ is a linear operator on $\widehat{\mathfrak{sv}}$ satisfying (5.9)-(5.13) for some $\epsilon \in \{\pm 1\}$, $\lambda \in \mathbb{Z}$, $a, d \in \mathbb{C}^*$, then $\tilde{\theta} \in Aut(\widehat{\mathfrak{sv}})$.* □

From the above lemmas and Theorem 5.5, we obtain the last main theorem.

**Theorem 5.10.** $Aut(\widetilde{\mathfrak{sv}}) \cong Aut(\widehat{\mathfrak{sv}})$.

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